Question 1.

(i) Let X be a well-ordered set. Show that every non-empty subset A of X has a greatest lower bound.

(ii) Give an example of an element $x \in S_\Omega$ (the first uncountable ordinal space) such that the singleton set $\{x\}$ has no strict greatest lower bound. (i.e. x has no immediate predecessor). Justify your answer.

Answer:

(i) Since X is a well-ordered set, every non-empty subset A of X has a smallest element, say x_A. Thus, $x \geq x_A$ for all $x \in A$. Therefore, x_A is a lower bound of A. Since $x_A \in A$, it follows that x_A is the greatest lower bound of A.

(ii) Let x be the smallest element of S_Ω. Then x has no immediate predecessor as it does not have any predecessor.

Question 2.

(i) Let A and B be subsets of a topological space X. Show that $\mathrm{Int}A \cup \mathrm{Int}B \subset \mathrm{Int}(A \cup B)$ where, for a subset A of X, $\mathrm{Int}A$ denotes the interior of A, defined by $\mathrm{Int}A = \cup\{V : V \text{ is open}, V \subset A\}$.

(ii) Give an example to show that the inclusion in part (i) above may be strict.

Answer:

(i) Let $x \in \mathrm{Int}A \cup \mathrm{Int}B$. Then, either $x \in \mathrm{Int}A$ or $x \in \mathrm{Int}B$. Without loss of generality, let $x \in \mathrm{Int}A = \cup\{V : V \text{ is open}, V \subset A\}$ then there exist an open set V in A such that $x \in V$. Now, $x \in V \subset A \subset A \cup B$ and V is also open in $A \cup B$. Therefore, $x \in \cup\{V : V \text{ is open}, V \subset A \cup B\} = \mathrm{Int}(A \cup B)$. This implies, $\mathrm{Int}A \cup \mathrm{Int}B \subset \mathrm{Int}(A \cup B)$.

(ii) Let $A = [0, 1]$, $B = [1, 2]$ and $X = \mathbb{R}$ with usual topology. Then $\mathrm{Int}A = (0, 1)$, $\mathrm{Int}B = (1, 2)$ and $\mathrm{Int}(A \cup B) = (0, 2)$. Therefore, in this case, $(0, 1) \cup (1, 2) = \mathrm{Int}A \cup \mathrm{Int}B \subsetneq \mathrm{Int}(A \cup B) = (0, 2)$

Question 3.

(i) Let $\mathbb{R}$ be given its usual order, and let $\mathbb{R} \times \mathbb{R}$ be given the dictionary order. Prove that the order topology from this dictionary order is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where the first factor $\mathbb{R}_d$ is the set of reals with the discrete topology, and the second factor is $\mathbb{R}$ with its usual topology.

(ii) Let X be a connected metric space containing at least two points. Show that X is uncountable.

Answer:

(i) The set $\{(a \times b, c \times d) | a, b \in \mathbb{R} \text{ with } (a < c) \text{ or } (a = c \text{ and } b < d)\}$ is a basis for dictionary topology $\mathcal{T}_1$ in $\mathbb{R} \times \mathbb{R}$, the set $\{\{a\} | a \in \mathbb{R}\}$ is a basis for $\mathbb{R}_d$ and the set $\{(a, b) | a, b \in \mathbb{R} \text{ and } a < b\}$ is a basis for $\mathbb{R}$ with usual topology. Therefore, the set $\{\{a\} \times (b, c) | a, b, c \in \mathbb{R} \text{ and } b < c\}$ is a basis for the product topology $\mathcal{T}_2 = \mathbb{R}_d \times \mathbb{R}$.

Proof of "$\mathcal{T}_1 \subset \mathcal{T}_2$": Let $x \times y \in \mathbb{R} \times \mathbb{R}$ and $(a \times b, c \times d)$ be an arbitrary basis element in $\mathcal{T}_1$ containing $x \times y$. Then there exists an open interval I in $\mathbb{R}$ containing y such that $x \times y \in \{x\} \times I \subset (a \times b, c \times d)$. Since $\{x\} \times I$ is a basis element in $\mathcal{T}_2$, by Lemma 13.3, it follows that $\mathcal{T}_1 \subset \mathcal{T}_2$.

Proof of "$\mathcal{T}_2 \subset \mathcal{T}_1$": This part is trivial as every basis element $\{a\} \times (b, c) \in \mathcal{T}_2$ is a basis element $\{a\} \times (b, c) = (a \times b, a \times c)$ in $\mathcal{T}_1$.

Therefore, the dictionary order topology in $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$.

(*ii*) Since X has at least two elements, let $a, b \in X$ such that $a \neq b$. Define $f : X \to [0, 1]$ by

$$f(x) := \frac{d(x,a)}{d(x,a) + d(x,b)}, \text{ for all } x \in X.$$

Then f is continuous as $x \to d(x, a)$ and $x \to d(x, b)$ are continuous with $d(x, a)+d(x, b) > 0$. Here $f(a) = 0$ and $f(b) = 1$. Since X is connected and the continuous image of connected space is connected, it follows that $[0, 1] \subset f(X)$, which shows that X is uncountable because $[0, 1]$ is uncountable.

Question 4.

(*i*) Show that there does not exist a continuous surjection $f : \mathbb{R} \to S_\Omega$, where $\mathbb{R}$ is the space of reals with usual topology and S_Ω is as in Question 1 above.

(*ii*) Let $K := \{1/n : n \in \mathbb{N}\}$ and let $\mathbb{R}_K$ denote the topological space given by the set of reals with the K-topology, which is generated by the basis:

$$\mathcal{B} := \{(a, b) : a, b \in \mathbb{R}\} \cup \{(a, b) \setminus K : a, b \in \mathbb{R}\}.$$

Show that $[0, 1]$ is not a compact subset of $\mathbb{R}_K$.

Answer:

(*i*) If possible, let $f : \mathbb{R} \to S_\Omega$ be a continuous surjection. Then S_Ω should be connected. Since S_Ω is a well-ordered set, it has a smallest element, say α. Let $\beta \in S_\Omega$ be such that $\alpha \neq \beta$. By definition of S_Ω, $S_\beta = \{x \in S_\Omega : x < \beta\} = [\alpha, \beta)$ is countable. Let γ be the immediate successor of α. Then, $(\alpha, \gamma) = \phi$. Therefore, $\{[\alpha, \gamma), (\alpha, \Omega)\}$ is a partition of S_Ω and hence S_Ω disconnected. This is a contradiction. Hence, there is no continuous surjection $f : \mathbb{R} \to S_\Omega$.

(*ii*) For each i, let $U_i = (\frac{1}{i}, 2) \cup (-1, 1) \setminus K$. Then $\{U_i\}_{i=1}^{\infty}$ is an open cover of $[0, 1]$. This open cover does not have a finite sub cover. So $[0, 1]$ is not compact subset of $\mathbb{R}_K$.